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HIGHER CONSERVATION LAW FOR THE MULTI-CENTRE METRICS

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Abstract

The multi-centre metrics are a family of euclidean solutions of the empty space Einstein equations with self-dual curvature. For this full class, we determine which metrics do exhibit an extra conserved quantity quadratic in the momenta, induced by a Killing-Stäckel tensor. Our results bring to light several metrics which correspond to classically integrable dynamical systems. They include, as particular cases, the Eguchi-Hanson and the Taub-NUT metrics.

1 Introduction

The discovery of the generalized Runge-Lenz vector for the Taub-NUT metric [7] has been playing an essential role in the analysis of its classical and quantum dynamics. As shown in [4] this triplet of conserved quantities gives quite elegantly the quantum bound states as well as the scattering states. The Killing-Stäckel tensors, which are the roots of the generalized Runge-Lenz vector, have been derived in [9] using purely geometric tools. As a result the classical integrability of the Taub-NUT metric was established. The other important metric of Eguchi-Hanson escaped to such an analysis (even though the results obtained in [13] suggested strongly classical integrability), to say nothing of the full family of the multi-centre metrics. It is the aim of this article to fill this gap.

In section 2 we have gathered a summary of known properties of the multi-centre metrics, their geodesic flow and some basic concepts about Killing-Stäckel and Killing-Yano tensors.

In section 3 we obtain the most general structure of the conserved quantity associated to a Killing-Stäckel tensor: it is a bilinear form in the momenta. Taking this quadratic structure as a starting point, we obtain the system of equations which ensure that such kind of a quantity is preserved by the geodesic flow. This system is analyzed and simplified. Its most important consequence is that the existence of an extra conserved quantity is related to the existence of an extra spatial Killing (besides the tri-holomorphic one), which may be either holomorphic or tri-holomorphic.

In section 4 we first consider the case of an extra spatial Killing which is holomorphic. We find that the extra conserved quantity does exist for the following families, with isometry $U(1) \times U(1)$:

1. The most general two-centre metric, with the potential

$$V = v_0 + \frac{m_1}{|\vec{r} + \vec{c}|} + \frac{m_2}{|\vec{r} - \vec{c}|},$$

which includes the double Taub-NUT metric for real $m_1 = m_2$ and the Eguchi-Hanson metric when we have in addition $v_0 = 0$. Our approach explains quite simply why there are three extra conserved quantities for Taub-NUT and only one for Eguchi-Hanson, and their very different nature.

2. A first dipolar breaking of Taub-NUT, with potential

$$V = v_0 + \frac{m}{r} + \mathcal{F} \frac{z}{r^3}.$$

In the Taub-NUT limit $\mathcal{F} \rightarrow 0$ the extra conserved quantity becomes trivial.

3. A second dipolar breaking of Taub-NUT with potential

$$V = v_0 + \frac{m}{r} + \mathcal{E} z.$$

In the Taub-NUT limit $\mathcal{E} \rightarrow 0$ there appears a triplet of extra conserved quantities: the generalized Runge-Lenz vector of [7].

The classical integrability of these three dynamical systems follows from our analysis.

In section 5 we consider the case of an extra spatial Killing which is tri-holomorphic. We find four different families of metrics, which share with the previous ones their classical integrability. Some conclusions are presented in section 6.

2 The Multi-Centre metrics

2.1 Background material

These euclidean metrics on M_4 have at least one Killing vector $\tilde{\mathcal{K}} = \partial_t$ and have the local form

$$g = \frac{1}{V} (dt + \Theta)^2 + V \gamma, \quad V = V(x), \quad \Theta = \Theta_i(x) dx^i, \quad (1)$$

where the x^i are the coordinates on γ . They are solutions of the empty space Einstein equations provided that :

1. The three dimensional metric γ is flat. Using cartesian coordinates x^i we can write

$$\gamma = d\vec{x} \cdot d\vec{x}. \quad (2)$$

2. Some monopole equation

$$dV = \eta *_{\gamma} d\Theta \quad (\eta = \pm 1). \quad (3)$$

Notice that the integrability condition for the monopole equation is $\Delta V = 0$, hence these metrics display an exact linearization of the empty space Einstein equations. They have been derived in many ways [12],[6],[10],[11]. In this last reference the geometric meaning of the cartesian coordinates x_i was obtained: they are nothing but the momentum maps of the complex structures under the circle action of ∂_t .

Let us summarize some background knowledge on the multi-centre metrics for further use. Taking for vierbein

$$E_a : \quad E_0 = \frac{1}{\sqrt{V}}(dt + \Theta), \quad E_i = \sqrt{V} dx_i$$

and defining as usual the spin connection Ω_{ab} and the curvature R_{ab} by

$$dE_a + \Omega_{ab} \wedge E_b = 0, \quad R_{ab} = d\Omega_{ab} + \Omega_{as} \wedge \Omega_{sb},$$

one can check that these metrics have a spin connection with η -self-duality:

$$\Omega_i^{(-\eta)} \equiv \Omega_{0i} - \frac{\eta}{2} \epsilon_{ijk} \Omega_{jk} = 0, \quad \implies \quad R_i^{(-\eta)} = 0,$$

which implies the η -self-duality of their curvature. It follows that they are hyperkähler and hence Ricci-flat.

The complex structures are given by the triplet of 2-forms

$$J_i = E_0 \wedge E_i - \frac{\eta}{2} \epsilon_{ijk} E_j \wedge E_k = (dt + \Theta) \wedge dx_i - \frac{\eta}{2} V \epsilon_{ijk} dx_j \wedge dx_k, \quad (4)$$

which are closed, in view of the hyperkähler property of these metrics.

Let us note that the self-duality of the complex structures and of the spin connection are opposite and that the Killing vector ∂_t is tri-holomorphic.

It is useful to define the Killing 1-form, dual of the vector $\tilde{\mathcal{K}} = \partial_t$, which reads

$$\mathcal{K} = \frac{dt + \Theta}{V}, \quad (5)$$

and plays some role in some in characterizing the multi-centre metrics.

Among these characterizations let us mention:

1. For the multi-centre metrics the differential $d\mathcal{K}$ has a self-duality opposite to that of the connection. A proof using spinors may be found in [14] and without spinors in [5].
2. The multi-centre metrics possess at least one tri-holomorphic Killing. For a proof see [9].

2.2 Geodesic flow

The geodesic flow is the Hamiltonian flow of the metric considered as a function on the cotangent bundle of M_4 . Using the coordinates (t, x_i) we will write a cotangent vector as

$$\Pi_i dx_i + \Pi_0 dt.$$

The symplectic form is then

$$\omega = dx_i \wedge d\Pi_i + dt \wedge d\Pi_0, \quad (6)$$

and we take for hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu = \frac{1}{2} \left(\frac{1}{V} (\Pi_i - \Pi_0 \Theta_i)^2 + V \Pi_0^2 \right). \quad (7)$$

For geodesics affinely parametrized by λ the equations for the flow allow on the one hand to express the velocities

$$\begin{aligned} \dot{t} &\equiv \frac{dt}{d\lambda} = \frac{\partial H}{\partial \Pi_0} = \left(V + \frac{\Theta^2}{V} \right) \Pi_0 - \frac{\Theta_i \Pi_i}{V}, \\ \dot{x}_i &\equiv \frac{dx_i}{d\lambda} = \frac{\partial H}{\partial p_i} = \frac{1}{V} p_i, \quad p_i = \Pi_i - \Pi_0 \Theta_i, \end{aligned} \quad (8)$$

and on the other hand to get the dynamical evolution equations

$$\begin{aligned} \dot{\Pi}_0 &= -\frac{\partial H}{\partial t} = 0, & \Pi_0 &= \frac{(\dot{t} + \Theta_i \dot{x}_i)}{V}, & (a) \\ \dot{\Pi}_i &= -\frac{\partial H}{\partial x_i} \implies \dot{p}_i = \left(\frac{H}{V} - q^2 \right) \partial_i V + \frac{q}{V} (\partial_i \Theta_s - \partial_s \Theta_i) p_s. & (b) \end{aligned} \quad (9)$$

Relation (9a) expresses the conservation of the charge $q = \Pi_0$, a consequence of the $U(1)$ isometry of the metric.

The conservation of the energy

$$H = \frac{1}{2} \left(\frac{p_i^2}{V} + q^2 V \right) = \frac{V}{2} (\dot{x}_i^2 + q^2) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (10)$$

is obvious since it expresses the constancy of the length of the tangent vector \dot{x}^μ along a geodesic.

For the multi-centre metrics, use of relation (3) brings the equations of motion to the nice form

$$\dot{\vec{p}} = \left(\frac{H}{V} - q^2 \right) \vec{\nabla} V + \eta \frac{q}{V} \vec{p} \wedge \vec{\nabla} V. \quad (11)$$

2.3 Killing-Stäckel versus Killing-Yano tensors

A Killing-Stäckel (KS) tensor is a symmetric tensor $S_{\mu\nu}$ which satisfies

$$\nabla_{(\mu} S_{\nu\rho)} = 0. \quad (12)$$

Let us observe that if K and L are two (possibly different) Killing vectors their symmetrized tensor product $K_{(\mu} L_{\nu)}$ is a KS tensor. So we will define *irreducible* KS tensors as the ones which cannot be written as linear combinations, with constant coefficients, of symmetrized tensor products of Killing vectors.

For a given KS tensor $S_{\mu\nu}$ the quantity

$$\mathcal{S} = S_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (13)$$

is preserved by the geodesic flow.

A Killing-Yano (KY) tensor is an antisymmetric tensor $Y_{\mu\nu}$ which satisfies

$$\nabla_{(\mu} Y_{\nu)\rho} = 0. \quad (14)$$

For instance a complex structure is an obvious KY tensor. One can build KS tensors from KY using:

Proposition 1 *If Y and Z are Killing-Yano tensors, then the tensor $S_{\mu\nu} = Y_\mu{}^\sigma Z_{\sigma\nu} + Z_\mu{}^\sigma Y_{\sigma\nu}$ is Killing-Stäckel.*

In [9] the triplet of KS tensors for the Taub-NUT metric was constructed, using proposition 1, from the known triplet of complex structures and a newly discovered KY tensor.

3 Quadratic conserved quantities

Instead of focusing ourselves on the KS tensor $S_{\mu\nu}$, whose usefulness is just to produce the conserved quantity \mathcal{S} , let us rather examine more closely the structure of the conserved quantity itself. To this end we expand relation (13) and get rid of the velocities using $\dot{t} = qV - \Theta_i \dot{x}_i$ and $\dot{x}_i = \frac{p_i}{V}$. This gives, for the conserved quantity we are looking for, the structure

$$\mathcal{S} = A_{ij}(x) p_i p_j + 2q B_i(x) p_i + C(x). \quad (15)$$

Our starting point will be to look for a conserved quantity having this particular property of being quadratic with respect to the momenta, forgetting that it was generated by a Killing-Stäckel tensor. Using the equations of the geodesic flow (8),(9) it is straightforward to impose that \mathcal{S} be conserved. One gets:

Proposition 2 *The quantity \mathcal{S} is conserved iff the following equations are satisfied*¹

$$\mathcal{L}_B V = 0 \quad (a)$$

$$\partial_{(k} A_{ij)} = 0 \quad (b)$$

$$\partial_{(i} B_{j)} - \eta A_{s(i} \epsilon_{j)su} \partial_u V = 0 \quad (c)$$

$$\partial_i C + 2(H - q^2 V) A_{is} \partial_s V - 2\eta q^2 \epsilon_{ist} B_s \partial_t V = 0 \quad (d)$$

(16)

¹We want to obtain quantities conserved for any values of q and H . So we have $H - q^2 V \neq 0$.

Relation (16a) shows that, to have some extra conservation law, we need an extra symmetry for the potential function V . This means that B_i must be conformal to some spatial Killing vector K_i , which is a symmetry of the potential V . Of course K_i lifts up to an isometry of the 4 dimensional metric. So we have obtained:

Proposition 3 *The number of extra conserved quantities of a multi-centre metric is at most equal to the number of extra spatial Killing vectors it does possess (besides the tri-holomorphic Killing $\tilde{K} = \partial_t$).*

Using this result we can discuss the triaxial generalization of the Eguchi-Hanson metric, with a tri-holomorphic $su(2)$, discovered in [1]. Its potential and cartesian coordinates were given in [8] in terms of the usual spherical coordinates. From these results it follows that this metric has no spatial Killing vector hence it will have no (irreducible) KS tensor.

For further analyses it is useful to define

$$B_i = -\eta F K_i. \quad (17)$$

The conserved quantity (15) becomes

$$\mathcal{S} = A_{ij}(x) p_i p_j - 2\eta q F K_i p_i + C(x), \quad (18)$$

and equation (16c) transforms into

$$K_{(i} \partial_{j)} F + A_{s(i} \epsilon_{j)su} \partial_u V = 0. \quad (19)$$

Taking its trace we see that $\mathcal{L}_K F = 0$, showing that V and F must have the same Killing.

3.1 Transformations of the system

Contracting (19) with $\partial_j V$ gives

Lemma 1 *The equation (19) has for consequence:*

$$(dV \cdot dF)K + \star(A[dV] \wedge dV) = 0. \quad (20)$$

We can proceed to:

Proposition 4 *The relation (19) is equivalent (except possibly at the points where the norm of the Killing K vanishes) to the relations:*

$$\begin{cases} A[K] = a(x) K, & a) \\ |K|^2 dF - A[\star(K \wedge dV)] + \star(A[K] \wedge dV) = 0 & b) \end{cases} \quad (21)$$

Proof : Contracting relation (19) with K_j gives relation b), while contracting with $K_i K_j$ we have

$$\epsilon_{stu} K_s A[K]_t \partial_u V = 0 \quad \implies \quad A[K]_i = a(x) K_i + b(x) \partial_i V, \quad (22)$$

which is not relation a). To complete the argument we first contract relation (19) with $\epsilon_{iab} K_a$; after some algebra we get

$$K_j \epsilon_{iab} \partial_i F K_a + 2A[K]_j \partial_b V + A[dV]_b K_j - K_s A[dV]_s \delta_{jb} - A_{ss} K_j \partial_b V = 0, \quad (23)$$

which, upon contraction with $A[K]_b$, gives eventually

$$(A[K]_s \partial_s V) A[K]_i = \{-\epsilon_{stu} K_s A[K]_t \partial_u F + A_{ss} A[K]_t \partial_t V - A[K]_s A[dV]_s\} K_i. \quad (24)$$

Let us now suppose that $A[K]_s \partial_s V \neq 0$. The previous relation shows that in (22) we must have $b(x) = 0$, hence $A[K]_s \partial_s V = 0$ which is a contradiction.

Let us prove that the converse is true. From (21b) we get

$$|K|^2 K_{(i} \partial_{j)} F + (K_{(j} A_{i)s} K_t \epsilon_{tsu} + A[K]_s K_{(j} \epsilon_{i)su}) \partial_u V = 0. \quad (25)$$

Use of the identity

$$A_{is} K_t K_j \epsilon_{tsu} \partial_u V = (|K|^2 A_{is} \epsilon_{jsu} - A[K]_i K_t \epsilon_{jtu}) \partial_u V \quad (26)$$

and of relation (21a) leaves us with (19), up to division by $|K|^2$. Notice that $|K|^2$ vanishes at the fixed points under the Killing action, i. e. in subsets of zero measure in \mathbb{R}^3 . ■

We can give, using (21a) and the identity

$$-A[\star(K \wedge dV)] = \star(A[K] \wedge dV) - A_{ss} \star(K \wedge dV) + \star(K \wedge A[dV]), \quad (27)$$

a simpler form to the relation (21b):

Lemma 2 *The relation (21b) is equivalent to*

$$|K|^2 dF + (2a - \text{Tr } A) \star(K \wedge dV) + \star(K \wedge A[dV]) = 0. \quad (28)$$

For further use let us prove:

Lemma 3 *To the spatial Killing K , leaving the potential V invariant, there corresponds a quantity Q invariant under the geodesic flow given by*

$$Q = K_i p_i + \eta q G, \quad \text{with} \quad i(K)F = -\eta dG. \quad (29)$$

Proof : We start from $\mathcal{L}_K V = 0$. Since K is a Killing we have $\mathcal{L}_K(\star dV) = \star d(\mathcal{L}_K V) = 0$, and (3) implies that $\mathcal{L}_K d\Theta = 0$. The closedness of $d\Theta$ implies $d(i(K)d\Theta) = 0$, and since our analysis is purely local in \mathbb{R}^3 , we can define

$$\eta dG = -i(K) d\Theta, \quad \implies \quad \star(K \wedge dV) = dG. \quad (30)$$

Then we multiply (9b) by p_i and get successively

$$K_i \dot{p}_i = (\dot{K}_i p_i) - \dot{K}_i p_i = (\dot{K}_i p_i) = \frac{q}{V} K_i (\partial_i \Theta_s - \partial_s \Theta_i) p_s = -\eta q \dot{x}_s \partial_s G = -\eta q \dot{G},$$

which concludes the proof. ■

Let us point out that if we use the coordinate ϕ adapted to the Killing $\tilde{K} = \partial_\phi$, we can write the connection $\Theta = \eta G d\phi$, where G does not depend on ϕ .

3.2 Integrability equations

We will derive now the integrability conditions for the equations (16c) and (16d). The first one was written using forms in (28) while the second one is

$$dC + 2(H - q^2 V)A[dV] + 2q^2 F \star (K \wedge dV) = 0. \quad (31)$$

It can now be proved :

Proposition 5 *The integrability condition for (31) is*

$$d A[dV] = 0 \quad \implies \quad A[dV] = dU \quad \text{and} \quad \mathcal{L}_K U = 0. \quad (32)$$

Proof : The integrability condition is obtained by differentiating (31). We get

$$2(H - q^2 V) d A[dV] + 2q^2 A[dV] \wedge dV + 2q^2 dF \wedge \star(K \wedge dV) + 2q^2 F d\star(K \wedge dV) = 0. \quad (33)$$

The last term in this equation vanishes in view of (30). Furthermore we have the identity specific to three dimensional spaces

$$dF \wedge \star(K \wedge dV) = -(K \cdot dF) \star dV + (dV \cdot dF) \star K = (dV \cdot dF) \star K$$

because K is a symmetry of F . Relation (33) simplifies to

$$2(H - q^2 V) d A[dV] + 2q^2 \star [(dV \cdot dF) K + \star(A[dV] \wedge dV)] = 0,$$

and lemma 1 implies the closedness of $A[dV]$. Since our analysis is purely local, the existence of U is a consequence of Poincaré's lemma.

The relations

$$\mathcal{L}_K U = i(K) dU = i(K) A[dV] = (A[K] \cdot dV) = a(K \cdot dV) = a \mathcal{L}_K V = 0$$

show the invariance of U under the Killing K . ■

Let us now turn to equation (28). We will prove:

Proposition 6 *The integrability condition for (28) is*

$$(2a - \text{Tr } A)dV + dU = |K|^2 \star d\tau, \quad \mathcal{L}_K d\tau = 0, \quad (34)$$

for some one form τ .

Proof : Let us define the 1-form

$$Y = (2a - \text{Tr } A)dV + dU. \quad (35)$$

It allows to write (28) and its integrability condition as

$$dF = -\star \left(\frac{K \wedge Y}{|K|^2} \right), \quad \delta \left(\frac{K \wedge Y}{|K|^2} \right) = 0, \quad (36)$$

or switching to components

$$K_i \delta \left(\frac{Y}{|K|^2} \right) + \frac{Y_s \partial_s K_i - K_s \partial_s Y_i}{|K|^2} = 0. \quad (37)$$

Let us examine the last terms. Since a and $\text{Tr } A$ are invariant under the Killing K , we obtain

$$Y_s \partial_s K_i - K_s \partial_s Y_i = -(2a - \text{Tr } A) \partial_i (K_s \partial_s V) - \partial_i (K_s \partial_s U) \quad (38)$$

and both terms vanish because V and U are invariant under K . We are left with the vanishing of the divergence of $Y/|K|^2$ from which we conclude (local analysis!) that it must have the structure $\star d\tau$ for some 1-form τ . From its definition it follows that $d\tau$ is invariant under K . ■

Using this result we can simplify (28) to

$$dF + \star(K \wedge \star d\tau) = dF - i(K)d\tau = 0. \quad (39)$$

Collecting all these results we have:

Proposition 7 *The quantity*

$$\mathcal{S} = A_{ij}(x) p_i p_j - 2\eta q F K_i p_i + C(x)$$

is preserved by the geodesic flow of the multi-centre metrics provided that the integrability constraints

$$\Delta V = 0, \quad A[dV] = dU, \quad (2a - \text{Tr } A) dV + dU = |K|^2 \star d\tau \quad (40)$$

and the following relations hold:

$$\begin{aligned} \mathcal{L}_K V &= 0, \\ \partial_{(k} A_{ij)} &= 0, \quad A[K] = a K, \\ dF &= i(K) d\tau, \\ d(C + 2HU) + 2q^2(-V dU + F dG) &= 0, \quad \star(K \wedge dV) = dG. \end{aligned} \quad (41)$$

3.3 Classification of the spatial Killing vectors

An important point, in view of classification, is whether the extra spatial Killing is tri-holomorphic or not. This can be checked thanks to:

Lemma 4 *The spatial Killing vector $K_i \partial_i$ is tri-holomorphic iff*

$$\epsilon_{ist} \partial_{[s} K_{t]} = 0.$$

Otherwise it is holomorphic.

Proof : From [2] we know that, for an hyperkähler geometry, a Killing may be either holomorphic or tri-holomorphic. As shown in [9] such a vector will be tri-holomorphic iff the differential of the dual 1-form $K = K_i dx_i$ has the self-duality opposite to that of the complex structures. A computation shows that this is equivalent to the vanishing of

$$dK^{(-\eta)} = -\frac{\eta}{2} \epsilon_{ijk} \partial_{[j} K_{k]} \left(E_0 \wedge E_i - \frac{\eta}{2} \epsilon_{ist} E_s \wedge E_t \right),$$

from which the lemma follows. ■

Since we are working in a flat three dimensional flat space, there are essentially two different cases to consider:

1. The Killing K generates a spatial rotation, which we can take, without loss of generality, around the z axis. In this case we have

$$K_i p_i = L_z$$

and this Killing vector is holomorphic with respect to the complex structure J_3 , defined in section 2.

2. The Killing K generates a spatial translation, which we can take, without loss of generality, along the z axis. In this case we have the

$$K_i p_i = p_z$$

and this Killing vector is tri-holomorphic.

We will discuss successively these two possibilities.

4 One extra holomorphic spatial Killing vector

One can get the general solution of the first equation for A_{ij} in (41b). It is most conveniently written in terms of $\mathcal{A}(p, p) \equiv A_{ij} p^i p^j$. One has:

$$\mathcal{A}(p, p) = \begin{cases} \alpha L_x^2 + \beta L_y^2 + \gamma L_z^2 + 2\mu L_y L_z + 2\nu L_z L_x + 2\lambda L_x L_y \\ + a_1 p_x L_y + a_2 p_x L_z + b_1 p_y L_x + b_2 p_y L_z + c_1 p_z L_x + c_2 p_z L_y \\ + d_1 p_x L_x + d_2 p_y L_y + a_{ij} p_i p_j. \end{cases} \quad (42)$$

At this point we have 20 free parameters. They reduce, when one imposes the existence of the rotational Killing, to

$$\mathcal{A}(p, p) = \alpha(L_x^2 + L_y^2) + \gamma L_z^2 + b(\vec{p} \wedge \vec{L})_z + a_{33} p_z^2 + a_{11} \vec{p}^2 + \delta p_z L_z. \quad (43)$$

We note that the parameter γ corresponds to a *reducible* piece which is just the square of L_z . We will take $\gamma = \alpha$ for convenience.

The parameter a_{11} is easily seen, upon integration of the remaining equations in (16), to give rise, in the conserved quantity \mathcal{S} , to the full piece

$$a_{11}(\vec{p}^2 - 2HV + q^2 V^2) \quad (44)$$

which vanishes thanks to the energy conservation (10). So we can take $a_{11} = 0$.

The second relation in (41b) implies the vanishing of δ . Hence, with slight changes in the notation, we end up with

$$\mathcal{A}(p, p) = a \vec{L}^2 + c^2 p_z^2 + b(\vec{p} \wedge \vec{L})_z. \quad (45)$$

Let us note that the parameters a and b are real while the parameter c may be either real or pure imaginary.

To take advantage of the rotational symmetry around the z axis we use first the coordinates $\rho = x^2 + y^2$ and z . From the system (41) one can check that the functions F and U are to be determined from

$$\begin{cases} F_{,\rho} = (az + b/2)V_{,z} - a/2 V_{,z} \\ F_{,z} = 2(az^2 + bz - c^2)V_{,\rho} - (az + b/2)V_{,z} \end{cases} \quad (46)$$

and

$$\begin{cases} U_{,\rho} = z(az + b)V_{,\rho} - \frac{1}{2}(az + b/2)V_{,z} \\ U_{,z} = -2\rho(az + b/2)V_{,\rho} + (a\rho + c^2)V_{,z} \end{cases} \quad (47)$$

We will write the connection

$$\Theta = \eta G d\phi, \quad x = \sqrt{\rho} \cos \phi, \quad y = \sqrt{\rho} \sin \phi. \quad (48)$$

Using lemma 3 we get the conserved quantity

$$J_z = L_z + \eta q G = x \Pi_y - y \Pi_x, \quad (49)$$

which will be useful in the integrability proof.

4.1 The two-centre metric

This case corresponds to the choice $a = 1$ and $c \neq 0$. Since $a = 1$, we can get rid of the constant b by a translation of the variable z . So, without loss of generality, we can take $b = 0$ and use the new variables $r_{\pm} = \sqrt{x^2 + y^2 + (z \pm c)^2}$. We get the relations

$$\partial_{r_+} F = -c \partial_{r_+} V, \quad \partial_{r_-} F = +c \partial_{r_-} V$$

which imply

$$V = f(r_+) + g(r_-), \quad F = -c(f(r_+) - g(r_-)).$$

Imposing to the potential V the Laplace equation we have

$$V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}, \quad F = -c \left(\frac{m_1}{r_+} - \frac{m_2}{r_-} \right) = -c\Delta, \quad (50)$$

i. e. we recover the most general 2-centre metric. Let us recall that the double Taub-NUT metric, given by real $m_1 = m_2$, is complete. If in addition we take the limit $v_0 \rightarrow 0$, we are led to the Eguchi-Hanson [3] metric.

One has then to check the integrability constraint (32) and to determine the functions U and C ²

$$U = -cz\Delta, \quad C = -2(H - q^2 V)U - q^2 r^2 \Delta^2, \quad r^2 = x^2 + y^2 + z^2. \quad (51)$$

Let us observe that the conserved quantity which we obtain may be real even if c is pure imaginary. In this case $m_1 = m$ may be complex, but if we take $m_2 = m^*$ the functions V and $c\Delta$ are real, as well as \mathcal{S} .

The final form of the conserved quantity for the two-centre metric is therefore

$$\begin{cases} \mathcal{S}_I = \vec{L}^2 + c^2 p_z^2 + 2\eta qc \Delta L_z + 2cz \Delta (H - q^2 V) - q^2 r^2 \Delta^2 \\ V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-} \quad \Delta = \frac{m_1}{r_+} - \frac{m_2}{r_-} \end{cases} \quad (52)$$

This conserved quantity is certainly different of the one exhibited in [9] since the latter does trivialize for the Eguchi-Hanson case while the former does not.

For completeness let us give the connection:

$$\Theta = \eta G d\phi, \quad G = m_1 \frac{z+c}{r_+} + m_2 \frac{z-c}{r_-}. \quad (53)$$

From the very definition of the coordinates r_{\pm} it is clear that the previous analysis is only valid for $c \neq 0$. The special case $c = 0$ will be examined now.

4.2 First dipolar breaking of Taub-NUT

This case corresponds to the choice $a = 1$ and $c = 0$. Since $a = 1$, we can again get rid of the parameter b . Then relation (46) for F implies

$$V = w_0(r) + w_1(r) z, \quad F_{,r} = -r w_1(r). \quad (54)$$

Imposing the Laplace equation we obtain

$$V = v_0 + \frac{m}{r} + \mathcal{E}z + \mathcal{F} \frac{z}{r^3}, \quad F = -\frac{\mathcal{E}}{2} r^2 + \frac{\mathcal{F}}{r}. \quad (55)$$

The integrability relations for U require that $\mathcal{E} = 0$ and we have

$$U = \mathcal{F} \frac{z}{r}, \quad C = -2\mathcal{F} \frac{z}{r} (H - q^2 V) - 2mq^2 \mathcal{F} \frac{z}{r^2} - q^2 \mathcal{F}^2 \frac{(3z^2 - r^2)}{r^4}. \quad (56)$$

The final form of the conserved quantity is therefore

$$\begin{cases} \mathcal{S}_{II} = \vec{L}^2 - 2\eta q \frac{\mathcal{F}}{r} L_z - 2\mathcal{F} \frac{z}{r} (H - q^2 v_0) + q^2 \mathcal{F}^2 \frac{(x^2 + y^2)}{r^4} \\ V = v_0 + \frac{m}{r} + \mathcal{F} \frac{z}{r^3} \end{cases} \quad (57)$$

The connection is:

$$\Theta = \eta G d\phi, \quad G = m \frac{z}{r} - \mathcal{F} \frac{x^2 + y^2}{r^3}. \quad (58)$$

Let us note that in the Taub-NUT limit ($\mathcal{F} \rightarrow 0$) the conserved quantity \mathcal{S}_{II} does become trivial.

²We discard constant terms in the function C .

4.3 Second dipolar breaking of Taub-NUT

This case corresponds to the choice $a = 0$ and $b = 1$. The relation (46) shows that by a translation of z we can take, without loss of generality, $c = 0$. From the integrability of F we deduce

$$V = f(r) + g(z), \quad F = \frac{1}{2}(f(r) - g(z)). \quad (59)$$

Imposing Laplace equation yields

$$V = v_0 + \frac{m}{r} + \mathcal{E}z, \quad F = \frac{1}{2} \left(\frac{m}{r} - \mathcal{E}z \right) \quad (60)$$

Then the integrability conditions for U are satisfied and we obtain

$$U = \frac{mz}{2r} - \frac{\mathcal{E}}{4}(x^2 + y^2), \quad C = -2U(H - q^2v_0) - 2q^2 m\mathcal{E} \frac{(x^2 + y^2)}{r}. \quad (61)$$

The final form of the conserved quantity is therefore

$$\left\{ \begin{array}{l} \mathcal{S}_{III} = (\vec{p} \wedge \vec{L})_z - \eta q \left(\frac{m}{r} - \mathcal{E}z \right) L_z - 2U(H - q^2v_0) - 2q^2 m\mathcal{E} \frac{(x^2 + y^2)}{r} \\ V = v_0 + \frac{m}{r} + \mathcal{E}z \quad U = \frac{mz}{2r} - \frac{\mathcal{E}}{4}(x^2 + y^2) \end{array} \right. \quad (62)$$

The gauge field Θ is given by

$$\Theta = \eta G d\phi, \quad G = m \frac{z}{r} + \frac{\mathcal{E}}{2}(x^2 + y^2). \quad (63)$$

For $\mathcal{E} = 0$ we are back to the Taub-NUT metric. In this case the spatial isometries are lifted up from $u(1)$ to $su(2)$. As a result we have now three possible Killings to start with

$$K_i^{(1)} p_i = L_x \quad K_i^{(2)} p_i = L_y \quad K_i^{(3)} p_i = L_z \quad (64)$$

and we expect that the conserved quantity found above should be part of a triplet. The two missing conserved quantities can be constructed following the same route which led to \mathcal{S}_{III} using the new available spatial Killings given by (64). We recover

$$\vec{\mathcal{S}} = \vec{p} \wedge \vec{L} - \eta q \frac{m}{r} \vec{L} + m(q^2v_0 - H) \frac{\vec{r}}{r}, \quad \mathcal{S}_{III}(\mathcal{E} = 0) \equiv \mathcal{S}_z. \quad (65)$$

Lemma 3 lifts up J_z , given by (49), to a triplet of conserved quantities

$$\vec{J} = \vec{L} + \eta q \frac{m}{r} \vec{r}, \quad (66)$$

which allows to write

$$\vec{\mathcal{S}} = \vec{p} \wedge \vec{J} + m(q^2v_0 - H) \frac{\vec{r}}{r}, \quad (67)$$

on which we recognize the generalized Runge-Lenz vector discovered by Gibbons and Manton [7].

We have therefore obtained, for the three hamiltonians $H_I, H_{II}(\mathcal{F} \neq 0)$ and H_{III} , corresponding respectively to the extra conserved quantities $\mathcal{S}_I, \mathcal{S}_{II}$ and \mathcal{S}_{III} , a set of four conserved quantities:

$$H, \quad q = \Pi_0, \quad J_z, \quad \mathcal{S},$$

which can be checked to be in involution with respect to the Poisson bracket.

Hence we conclude to:

Proposition 8 *The three hamiltonians H_I , $H_{II}(\mathcal{F} \neq 0)$ and H_{III} , defined above are integrable in Liouville sense.*

This includes, as a special case, the classical integrability of the Eguchi-Hanson metric.

5 One extra tri-holomorphic spatial Killing vector

This time we have for Killing $K_i p_i = p_z$. Imposing this translational invariance and the constraint $A[K] \propto K$ restricts $\mathcal{A}(p, p)$ to have the form

$$\mathcal{A}(p, p) = a L_z^2 - 2b p_x L_z + 2c p_y L_z + \sum_{i,j=1}^2 a_{ij} p_i p_j. \quad (68)$$

We have omitted a term proportional to p_z^2 since it is reducible.

The functions F and U , which depend only on the coordinates x and y , using the system (41), are seen to be determined by

$$\begin{cases} F_{,x} = A_{12} V_{,x} - A_{11} V_{,y} \\ F_{,y} = A_{22} V_{,x} - A_{12} V_{,y} \end{cases} \quad \begin{cases} U_{,x} = A_{11} V_{,x} + A_{12} V_{,y} \\ U_{,y} = A_{12} V_{,x} + A_{22} V_{,y} \end{cases} \quad (69)$$

with

$$A_{11} = ay^2 + 2by + a_{11}, \quad A_{22} = ax^2 + 2cx + a_{22}, \quad A_{12} = -axy - bx - cy + a_{12}. \quad (70)$$

The connection and the conserved quantity related to p_z will be

$$\Theta = \eta G dz, \quad \Pi_z = p_z + \eta q G. \quad (71)$$

In order to organize the subsequent discussion, let us observe:

1. For $a \neq 0$, we may take $a = 1$. The spatial translations allow to take $b = c = 0$, and a rotation $a_{12} = 0$ as well. Hence we are left with

$$\mathcal{A}(p, p) = L_z^2 + (a_{11} - a_{22}) p_x^2 + a_{22}(p_x^2 + p_y^2).$$

Adding the reducible term $a_{22} p_z^2$ we recover the piece $a_{22} \vec{p}^2$ which can be discarded, as already explained in section 4. So we will take for our first case

$$\mathcal{A}_1(p, p) = L_z^2 - c^2 p_x^2, \quad c \in \mathbb{R} \cup i\mathbb{R}, \quad c \neq 0. \quad (72)$$

2. Our second case, which is the singular limit $c \rightarrow 0$ of the first case, corresponds to

$$\mathcal{A}_2(p, p) = L_z^2. \quad (73)$$

3. For $a = 0$, a first translation allows to take $a_{12} = 0$, while the second one allows the choice $a_{11} = a_{22}$ and the corresponding term $a_{11}(p_x^2 + p_y^2)$ is disposed of as in the first case. Eventually a rotation will bring b to zero and $c = 1$. Our third case will be

$$\mathcal{A}_3(p, p) = p_y L_z. \quad (74)$$

4. For $a = b = c = 0$, a rotation brings a_{12} to zero. Discarding $p_x^2 + p_y^2$, we are left with our fourth case

$$\mathcal{A}_4(p, p) = p_x^2 - p_y^2. \quad (75)$$

We will state the results obtained for these four cases without going through the detailed computations, which are greatly simplified using the complex coordinates $\zeta = x + iy$ and $\bar{\zeta} = x - iy$.

5.1 First case

Writing the corresponding metric

$$\frac{1}{V}(dt + \tau dz)^2 + V(dz^2 + d\bar{\zeta} d\zeta), \quad (76)$$

and the conserved quantity as

$$\mathcal{S}_1 = L_z^2 - c^2 p_x^2 - 2\eta q F \Pi_z + 2(q^2 v_0 - H) U + q^2 D, \quad c \neq 0, \quad (77)$$

we have

$$\begin{aligned} \bullet \quad V &= v_0 + m \frac{\zeta}{\sqrt{\zeta^2 + c^2}} + \bar{m} \frac{\bar{\zeta}}{\sqrt{\bar{\zeta}^2 + c^2}}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C} \\ \bullet \quad V - v_0 + i G &= 2m \frac{\zeta}{\sqrt{\zeta^2 + c^2}}, \quad U + i F = -mc^2 \frac{\zeta + \bar{\zeta}}{\sqrt{\zeta^2 + c^2}}, \\ \bullet \quad D &= -2c^2 |m|^2 \frac{(\zeta^2 + \bar{\zeta}^2 + |\zeta|^2 + c^2)}{|\zeta^2 + c^2|}. \end{aligned} \quad (78)$$

If one is willing to use spheroidal coordinates ξ and η defined (for $c^2 > 0$) by

$$x = \frac{1}{c} \sqrt{(\xi^2 - c^2)(c^2 - \eta^2)} \quad y = \frac{1}{c} \xi \eta,$$

it is possible to write the results in terms of real quantities. This has the drawback that the cases $c^2 > 0$ and $c^2 < 0$ need a separate analysis.

5.2 Second case

Writing the conserved quantity as

$$\mathcal{S}_2 = L_z^2 - 2\eta q F \Pi_z + 2(q^2 v_0 - H) U, \quad (79)$$

we have:

$$\begin{aligned} \bullet \quad V &= v_0 + \frac{m}{\zeta} + \frac{\bar{m}}{\bar{\zeta}}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C}, \\ \bullet \quad V - v_0 + i G &= 2 \frac{m}{\zeta}, \quad U + i F = 2m \frac{\bar{\zeta}}{\zeta}. \end{aligned} \quad (80)$$

5.3 Third case

Writing the conserved quantity as

$$\mathcal{S}_3 = p_y L_z - 2\eta q F \Pi_z + 2(q^2 v_0 - H) U + q^2 D, \quad (81)$$

we have:

$$\begin{aligned} \bullet \quad & V = v_0 + \frac{m}{\sqrt{\zeta}} + \frac{\bar{m}}{\sqrt{\bar{\zeta}}}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C}, \\ \bullet \quad & V - v_0 + i G = 2 \frac{m}{\sqrt{\zeta}}, \quad U + i F = \frac{m}{2} \frac{\bar{\zeta} - \zeta}{\sqrt{\zeta}} \\ \bullet \quad & D = |m|^2 \left(\sqrt{\frac{\bar{\zeta}}{\zeta}} + \sqrt{\frac{\zeta}{\bar{\zeta}}} \right) \end{aligned} \quad (82)$$

5.4 Fourth case

Writing the conserved quantity as

$$\mathcal{S}_4 = p_x^2 - p_y^2 - 2\eta q F \Pi_z + 2(q^2 v_0 - H) U + q^2 D \quad (83)$$

we have

$$\begin{aligned} \bullet \quad & V = v_0 + m \zeta + \bar{m} \bar{\zeta}, \quad v_0 \in \mathbb{R}, \quad m \in \mathbb{C}, \\ \bullet \quad & V - v_0 + i G = 2m \zeta, \quad U + i F = 2m \bar{\zeta}, \\ \bullet \quad & D = 2|m|^2(\zeta^2 + \bar{\zeta}^2). \end{aligned} \quad (84)$$

As was the case when the extra spatial Killing was holomorphic, we have obtained for the four hamiltonians considered in this section, a set of four conserved quantities

$$H, \quad q = \Pi_0, \quad \Pi_z, \quad \mathcal{S},$$

which are in involution with respect to the Poisson bracket, hence we conclude to:

Proposition 9 *The four hamiltonians determined in this section are integrable in Liouville sense.*

Let us conclude with two remarks:

1. One should notice that, among the four potentials considered in this section, only the second one and the fourth one are *uniform* functions in the three dimensional flat space.

2. The fourth case analyzed in this section is also interesting because it may exhibit *super-integrability*, i. e. more than four conserved quantities. According to the choice of the parameter m we have one more spatial Killing and one more conserved quantity

$$m = \bar{m} \implies \{\partial_y, \Pi_y\} \quad ; \quad m = -\bar{m} \implies \{\partial_x, \Pi_x\}. \quad (85)$$

The algebra generated by these five conserved quantities, with respect to the Poisson bracket, remains fully abelian. This phenomenon of super-integrability, which is of quite common experience in classical mechanics, seems quite rare within the multi-centre, since it is exhibited only by one family out of seven.

6 Conclusion

We have settled the problem of finding all the multi-centre metrics which do exhibit some extra quantity, quadratic with respect to the momenta, and preserved by the geodesic flow. The concept of Killing-Stäckel can be generalized to of type $(n, 0)$, with $n \geq 3$. Such a tensor has to be fully symmetric and such that

$$\nabla_{(\lambda} S_{\mu_1 \dots \mu_n)} = 0.$$

It follows that the geodesic flow preserves the quantity

$$S_{\mu_1 \dots \mu_n} \dot{x}^{\mu_1} \dots \dot{x}^{\mu_n}.$$

The corresponding invariants will be cubic, quartic, etc... with respect to the momenta. Little is known about the existence of such objects for the multi-centre metrics.

Let us put emphasis also on the purely local nature of our analysis: it makes no difference between complete and non-complete metrics. For instance in section 4 we have seen that the most general two-centre metric is integrable, however it is complete only for real $m_1 = m_2$, i. e. for the double Taub-NUT metric.

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